

A Generic Test of Modified Gravity Models which Emulate Dark Matter

E. O. Kahya* and R. P. Woodard†

Department of Physics, University of Florida, Gainesville, FL 32611, USA

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We propose a generic test for models in which gravity is modified to do away with dark matter. These models tend to have gravitons couple to a different metric than ordinary matter. A strong test of such models comes from comparing the arrival time of the gravitational wave pulse from a cosmological event such as a supernova with the arrival times of the associated pulses of neutrinos and photons. For SN 1987a we show that the gravity wave would have arrived 5.3 days after the neutrino pulse.

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1. Introduction

As early as 1933 Zwicky was able to infer the need for dark matter by applying the virial theorem to observations of the Coma Cluster [1]. In the 1970's Rubin, Ford and Thonnard accumulated independent evidence from the rotation curves of spiral galaxies [2, 3, 4]. And the 1990's saw weak lensing used to probe dark matter in galactic clusters [5, 6, 7, 8, 9, 10].

It will be seen that the evidence for dark matter has so far been entirely restricted to gravity. That is, one infers the Einstein tensor from measurements of cosmic motions, or from lensing, and then compares the result with the observed stress-energy in stars and gas,

$$\left(R_{\mu\nu} - \frac{1}{2}g_{\mu\nu}R\right)_{\text{inf}} = \frac{8\pi G}{c^4} \left(T_{\mu\nu}\right)_{\text{obs}}. \quad (1)$$

There is indeed a disagreement between the two sides of the equation, but it is not clear that this signifies the need for dark matter on the right of (1) rather than a modification of gravity on the left. Although there are plausible dark matter candidates, none of them has so far been detected in a terrestrial laboratory [11, 12, 13].

Certain regularities in cosmic structures suggest modified gravity. One of these is the Tully-Fisher relation, which states that the luminosity of a spiral galaxy is proportional to the fourth power of the peak velocity in its rotation curve [14]. If luminous matter is insignificant compared to dark matter, why should such a relation exist? Another regularity is Milgrom's Law, which states that the need for dark matter occurs at gravitational accelerations of $a_0 \simeq 10^{-10} \text{ m/s}^2$ [15]. This has been observed in cosmic structures which vary in size by six orders of magnitude [16].

A modification of Newtonian gravity which explains these regularities was proposed by Milgrom in 1983 [17]. His model, Modified Newtonian Dynamics (MOND), was soon given a Lagrangian formulation in which conservation of energy, 3-momentum and angular momentum are

manifest [18]. However, there was for years no successful relativistic generalization which could be employed to study cosmological evolution. Even in the context of static, spherically symmetric geometries,

$$ds^2 \equiv -B(r)c^2dt^2 + A(r)dr^2 + r^2d\Omega^2, \quad (2)$$

the early formulation of MOND fixed only $B(r)$, not $A(r)$. It was therefore incapable of making definitive predictions about gravitational lensing.

A relativistic extension of MOND has recently been proposed by Bekenstein [19]. This model is known as TeVeS for "Tensor-Vector-Scalar." In addition to reproducing the MOND force law at low accelerations, TeVeS has acceptable post Newtonian parameters, and it gives a plausible amount of gravitational lensing [19]. When TeVeS is used in place of general relativity + dark matter to study cosmological evolution, the results are in better agreement with data than many thought possible [20, 21, 22, 23].

What concerns us here is the curious property of TeVeS that small amplitude gravity waves are governed by general relativity through the metric $g_{\mu\nu}$, whereas matter couples to a "disformally transformed" metric which involves the vector and scalar fields,

$$\tilde{g}_{\mu\nu} = e^{-2\phi}(g_{\mu\nu} + A_\mu A_\nu) - e^{2\phi}A_\mu A_\nu. \quad (3)$$

The Scalar-Vector-Tensor gravity (SVTG) theory proposed by Moffat also has different metrics for matter and small amplitude gravity waves [24, 25]. The appearance of this feature in two very different models is the result of trying to reconcile solar system tests with modified gravity at ultra-low accelerations. Solar system tests strongly predispose the Lagrangian to possess an Einstein-Hilbert term [26]. On the other hand, failed attempts to generalize MOND [27] have led to a theorem that one cannot get sufficient weak lensing from a stable, covariant and purely metric theory which reproduces the Tully-Fisher relation without dark matter [28]. Hence the MOND

force must be carried by some other field, and it is a combination of this other field and the metric which determines the geodesics for ordinary matter. However, the dynamics of small amplitude gravity waves are still set by the linearized Einstein equation. This simple observation makes for a sensitive and generic test.

We define a *Dark Matter Emulator* as any modified gravity theory for which:

1. Ordinary matter couples to the metric $\tilde{g}_{\mu\nu}$ that would be produced by general relativity + dark matter; and
2. Small amplitude gravity waves couple to the metric $g_{\mu\nu}$ produced by general relativity without dark matter.

Now consider a cosmic event such as a supernova which emits simultaneous pulses of gravity waves and either neutrinos or photons. If physics is described by a dark matter emulator then the pulse of gravity waves will reach us on a lightlike geodesic of $g_{\mu\nu}$, whereas neutrinos and photons travel along a lightlike geodesic of $\tilde{g}_{\mu\nu}$. If significant propagation occurs over regions that would be dark matter dominated in general relativity then there will be a measurable lag between arrival times.

The magnitude of the expected time lag is so large that great precision is not needed for either $g_{\mu\nu}$ or $\tilde{g}_{\mu\nu}$. We demonstrate this in section 2 by computing the time lag for SN 1987a using simple models for the distribution of luminous and dark matter. Section 3 discusses the prospects for observation. Our conclusions comprise section 4.

2. Time Lag

We model the luminous matter of our galaxy as a spherical bulge of $M \simeq 10^{41}$ kg. For this source general relativity gives a metric of the form (2), with the functions B and A expressed in terms of twice the Newtonian potential $\epsilon \equiv (2GM)/(c^2 r)$,

$$g_{\mu\nu} \implies B(r) = 1 - \epsilon, \quad A(r) = \frac{1}{1 - \epsilon} \simeq 1 + \epsilon. \quad (4)$$

At the Sun's radius of $r_S \simeq 8.0$ kpc we have $\epsilon \simeq 6 \times 10^{-7}$, so it is valid to neglect terms of order ϵ^2 .

To determine $\tilde{g}_{\mu\nu}$ we assume the galaxy is surrounded by an isothermal halo which makes the asymptotic rotation speed $v_* = (a_0 GM)^{1/4}$. This gives rise to a second small parameter, $\epsilon_* \equiv 2v_*^2/c^2 \simeq 6 \times 10^{-7}$. For this source general relativity gives a metric of the form (2) with,

$$\tilde{g}_{\mu\nu} \implies \tilde{B}(r) \simeq 1 - \epsilon + \epsilon_* \ln\left(\frac{r}{r_S}\right), \quad \tilde{A}(r) \simeq 1 + \epsilon + \epsilon_*. \quad (5)$$

We have chosen the integration constant in the order ϵ_* contribution to $\tilde{B}(r)$ so that it vanishes at $r = r_S$. In section 4 we discuss the effect of other choices.

It is most efficient to work this problem in Cartesian coordinates for which the nonzero components of the two metrics are,

$$g_{00} = -1 + \epsilon, \quad g_{ij} \simeq \delta^{ij} + \epsilon \hat{r}^i \hat{r}^j, \quad (6)$$

$$\tilde{g}_{00} \simeq -1 + \epsilon + \epsilon_* \ln\left(\frac{r}{r_S}\right), \quad \tilde{g}_{ij} \simeq \delta^{ij} + (\epsilon + \epsilon_*) \hat{r}^i \hat{r}^j. \quad (7)$$

Here the radial unit vector is $\hat{r}^i \equiv x^i/r$, the radius is $r = \|\vec{x}\|$ and we have neglected terms which are higher than linear in either ϵ or ϵ_* .

In each metric we must construct the lightlike geodesic from a point \vec{x}_L in the Large Magellanic Cloud to the Sun's position \vec{x}_S . For $g_{\mu\nu}$ this means solving the second order equation,

$$\ddot{\chi}^\mu(\tau) + \Gamma_{\rho\sigma}^\mu(\chi(\tau)) \dot{\chi}^\rho(\tau) \dot{\chi}^\sigma(\tau) = 0. \quad (8)$$

The general solution to (8) depends upon eight integration constants which can be taken as the initial position $\chi^\mu(0)$ and velocity $\dot{\chi}^\mu(0)$. Because $g_{\mu\nu}$ is static we may as well start the geodesic at $t = 0$,

$$\chi^0(0) = 0, \quad \chi^i(0) = x_L^i. \quad (9)$$

We fix the spatial components of the initial velocities by requiring the geodesic to reach the Sun at $\tau = 1$,

$$\chi^i(1) = x_S^i \iff \dot{\chi}^i(0). \quad (10)$$

The initial temporal velocity is fixed by requiring the geodesic to be lightlike,

$$g_{\mu\nu}(0, \vec{x}_L) \dot{\chi}^\mu(0) \dot{\chi}^\nu(0) = 0 \iff \dot{\chi}^0(0). \quad (11)$$

The geodesic $\tilde{\chi}^\mu(\tau)$ is determined the same way, but with the substitutions of $\tilde{\Gamma}_{\rho\sigma}^\mu$ for $\Gamma_{\rho\sigma}^\mu$ in (8) and of $\tilde{g}_{\mu\nu}$ for $g_{\mu\nu}$ in (11). The time lag we seek is,

$$\Delta t \equiv \frac{1}{c} (\tilde{\chi}^0(1) - \chi^0(1)). \quad (12)$$

Because ϵ and ϵ_* are so small we can use perturbation theory. The 0th order (flat space) solution is the same for both geodesics,

$$\bar{\chi}^0(\tau) = \Delta x \tau, \quad \bar{\chi}^i = x_L^i + \Delta x^i \tau, \quad (13)$$

where $\Delta x^i \equiv x_S^i - x_L^i$. Because both $g_{\mu\nu}$ and $\tilde{g}_{\mu\nu}$ have the same order ϵ terms, the order ϵ corrections to χ^μ cancel those to $\tilde{\chi}^\mu$ in (12) and we need only consider the corrections of order ϵ_* . The nonzero components of the order ϵ_* connection are,

$$\Delta \Gamma_{0i}^0 = \Delta \Gamma_{00}^i = \frac{\epsilon_* \hat{r}^i}{2r}, \quad \Delta \Gamma_{jk}^i = \frac{\epsilon_* \hat{r}^i}{r} [\delta^{jk} - \hat{r}^j \hat{r}^k]. \quad (14)$$

The 1st order corrected geodesic follows from integrating the geodesic equation,

$$\Delta \chi^\mu(\tau) \equiv \tilde{\chi}^\mu(\tau) - \chi^\mu(\tau), \quad (15)$$

$$= \Delta \dot{\chi}^\mu(0) - \int_0^\tau d\tau' \int_0^{\tau'} d\tau'' \Delta \Gamma_{\rho\sigma}^\mu(\bar{\chi}(\tau'')) \dot{\bar{\chi}}^\rho \dot{\bar{\chi}}^\sigma. \quad (16)$$

The spatial boundary conditions (10) imply,

$$\Delta\dot{\chi}^i(0) = \int_0^1 d\tau (1-\tau) \Delta\Gamma^i_{\rho\sigma}(\bar{\chi}(\tau)) \dot{\bar{\chi}}^\rho \dot{\bar{\chi}}^\sigma. \quad (17)$$

The lightlike condition (11) fixes the final integration constant,

$$\Delta\dot{\chi}^0(0) = \frac{\Delta x^i \Delta\dot{\chi}^i(0)}{\Delta x} + \frac{\epsilon_* \Delta x}{2} \ln\left(\frac{r_L}{r_S}\right) + \frac{\epsilon_*(\hat{r} \cdot \Delta\vec{x})^2}{2\Delta x}. \quad (18)$$

The parameter integrations are straightforward when expressed in terms of the dimensionless constants,

$$\alpha \equiv \frac{\vec{x}_L \cdot \Delta\vec{x}}{\Delta x^2} \quad \text{and} \quad \beta \equiv \frac{r_L^2}{\Delta x^2}. \quad (19)$$

Our result for the first order time lag is,

$$\Delta t = \frac{\epsilon_* \Delta x}{c} \left[1 + \frac{\alpha}{2} \ln\left(\frac{r_L}{r_S}\right) - \sqrt{\beta - \alpha^2} \tan^{-1}\left(\frac{\sqrt{\beta - \alpha^2}}{\beta + \alpha}\right) \right]. \quad (20)$$

Within the context of our simple model for the matter distributions, equation (20) is valid for any \vec{x}_L . The specific values appropriate to SN 1987a are [29],

$$r_S \simeq 8.0 \text{ kpc} \quad , \quad r_L \simeq 50.9 \text{ kpc} \quad , \quad \Delta x \simeq 51.4 \text{ kpc} \quad , \\ \alpha \simeq -.9775 \quad \text{and} \quad \beta \simeq .9793. \quad (21)$$

Substituting these values in (20) gives,

$$\Delta t \Big|_{\text{SN 1987a}} \simeq 36.7 \text{ days} \left[-.144 \right] \simeq -5.3 \text{ days}. \quad (22)$$

Therefore simultaneous pulses of neutrinos and/or photons would have arrived 5.3 days before the gravity waves if physics were described by a dark matter emulator.

The sign of (20) is mostly set by the difference of the two gravitational potentials, $\tilde{B}(r) - B(r) \simeq \epsilon_* \ln(r/r_S)$. The lag is negative because ordinary matter is at higher potential than gravity waves. The square bracketed term in (20) is typically of order one. (It is $-.144$ for SN 1987a.) Hence the rough magnitude of the time lag is $\epsilon_* \simeq 6 \times 10^{-7}$ times the flat space propagation time. This works out to be 36.7 days for SN 1987a. One could therefore anticipate a lag in the range of days without the detailed computation.

3. Observational Prospects

The test we have proposed requires detecting gravity waves and either neutrinos or photons that were emitted simultaneously from some cosmic event separated from us by an extensive region in which the gravitational influence of dark matter predominates over that of luminous

matter. If physics is described by general relativity + dark matter then the arrival times will be simultaneous. If physics is instead described by a dark matter emulator then the photon and neutrino pulses will generally arrive some days before the gravity wave signal.

One only expects a few supernovae per century in our galaxy. Of course the light signal from a supernova can be observed even from distant galaxies, but this is much further than we have any hope of detecting neutrinos or gravity waves. The optical signal from a supernova will typically lag the neutrino pulse by several hours — as was the case for SN 1987a [29] — because photons must traverse the optically dense stellar environment. However, the magnitude of the expected delay for the gravity wave signal is so great that this should not matter except for nearby supernovae.

The amount of gravitational radiation from a supernova depends on the oblateness of the progenitor. If the oblateness of SN 1987a was in relation to that of the Sun then current gravitational wave detectors would probably not have seen anything [30]. However, advanced LIGO would detect such a supernova out to 0.8 Mpc [30]. This includes the Andromeda galaxy, which doubles the expected rate and also ensures that the signal passes through dark matter dominated regions.

Neutrinos from SN 1987a were observed by the Kamiokande-II [31, 32] and Irvine-Michigan-Brookhaven [33, 34] detectors. Of course the state of the art has improved since then, although effective coverage is still limited to our own galaxy and its satellites [35, 36]. There is now a Supernova Early Warning System [37] comprised of Super-Kamiokande [38], the Large Volume Detector [39], the Sudbury Neutrino Observatory [40] — which is no longer operating but may be succeeded by SNO+ [41] — and the IceCube Detector [42].

4. Discussion

Our largest uncertainty is modeling the radius at which the density of dark matter reaches its asymptotic form of $\rho(r) \simeq v_*^2/(4\pi G r^2)$. We chose that radius to be r_S in (5) because this is near where the Newtonian acceleration reaches the value of $a_0 \simeq 10^{-10} \text{ m/s}^2$ at which Milgrom's Law implies the need for dark matter [15]. Had we chosen some other radius, r_O , the square bracketed term in (20) would change to

$$\left[\quad \right] \longrightarrow \left[\quad \right] - \frac{1}{2} \ln\left(\frac{r_S}{r_O}\right). \quad (23)$$

One would expect this to make the time lag more negative because it is difficult to imagine having dark matter “turn on” much beyond r_S .

If neutrinos have mass then a neutrino of energy E

would propagate with velocity,

$$v = c\sqrt{1 - \frac{m^2 c^4}{E^2}}. \quad (24)$$

Taking the average detected supernova neutrino energy to be $E \simeq 10$ MeV [43], and the (electron) neutrino mass to be its current upper limit of $m \simeq 2$ eV [44], the velocity would be changed by $\Delta v \simeq -2 \times 10^{-14}c$. Over a distance of 50 kpc this would delay the neutrino by .1 s, which is negligible on the scale of days expected for the primary effect.

The generality of our test derives from avoiding the details of specific models and concentrating instead upon what they do, which is to emulate dark matter. Rotation curves and lensing are too well measured for any viable model not to somehow reproduce the metric $\tilde{g}_{\mu\nu}$ which couples to ordinary matter. We have explained the theoretical reasons why gravity waves are predisposed to couple to the original metric $g_{\mu\nu}$ [26, 28]. Both of the existing models do have this property [19, 24], but as we cannot prove it to be unavoidable, it is safest to comment that our results are limited to dark matter emulators.

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* kahya@phys.ufl.edu

† woodard@phys.ufl.edu

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